LAHORE
SCIENCE
FOUNDRY

مشاہدہ
phenomenon lab

Phenomenon# 12 مشاہدہ#۱۲

# The Song of Fluids

## سیالات کی سرگوشیاں

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The four fundamental states of matter are solid, liquid, gas, and plasma. Of these three—liquid, gas, and plasma—are fluids. We breathe in fluids, drink fluids, and are, in fact, constantly surrounded by them. The defining characteristic of fluids is that they can "flow" and move freely, leading to fascinating and even chaotic observations. Today, we are going to explore many such observations. You will get to see how a simple drop of liquid can move in surprising ways, and you might even discover some unexpected results!

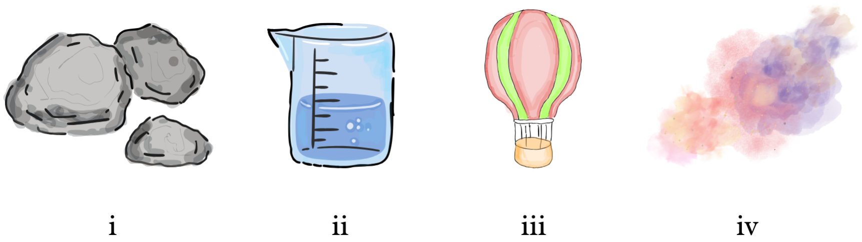
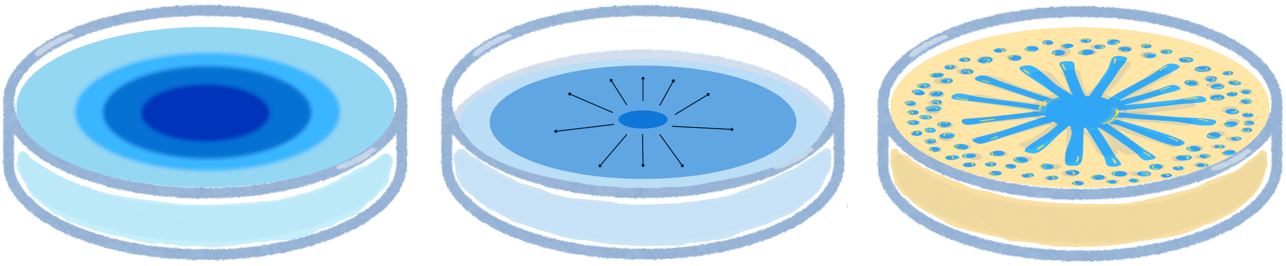


Figure 1: i: solid rocks, ii: liquid in beaker, iii: gas in hot air balloon, iv: plasma in space.

For those of you joining this exploration, here's what to expect. This document is divided into two sections, with the first further split into two parts. We start with diffusion (Section 1a) to understand how particles move and spread. Then, in Marangoni flows (Section 1b), we explore how surface tension creates motion in liquids. Finally, in Marangoni bursting (Section 2), we see how these effects lead to something even more dramatic. Along the way, you'll analyze videos, collect data, and compare results, breaking everything down step by step.



(i) Shallow Water (ii) Ultra-shallow Water (iii) Shallow Oil

Figure 2: Fluid interactions in different conditions, as explored later in this document.

# 1a) Ink in Water: Diffusion

One behavior of fluids that you may have already studied is diffusion. It's the process where particles move from a region of high concentration to a region of low concentration due to constant molecular motion. But why does this happen? The movement isn't smooth or predictable. Instead, particles follow a chaotic, zigzagging path as they collide with surrounding molecules. This phenomenon is known as Brownian motion. Diffusion happens all around us. It's how oxygen reaches your cells, how scent spreads in a room, and how heat moves through the air. From the tiniest molecules in your body to gases in the atmosphere, it's always happening, always following the same rules.

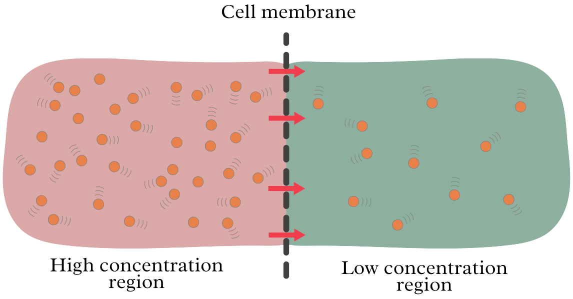


Figure 3: Diffusion in cells through the cell membrane.

### *But if diffusion is random, how can it still follow rules?*

Particles in a fluid move randomly, bumping into each other, taking unpredictable steps in all directions. This is called a **random walk**. If we followed just one, we'd have no idea where it would end up. But with thousands, even millions of them, a pattern starts to appear. Instead of tracking each one, we use mathematical and statistical tools to describe how they spread over time. So while every step is random, the overall movement follows clear rules—and that's where **Fick's Laws** come in.

## Fick's First Law: Why do particles move?

If one region has more particles than another, there will be a *net movement from high concentration to low concentration.* This happens because, on average, more particles step into the less crowded side than the other way around.

We can describe this using a simple idea:

$$\textbf{\textit{Flux}} \propto \textbf{\textit{Change in concentration over distance}}$$

That is, the movement of particles is *proportional* to how unevenly spread out they are. The steeper the concentration gradient, the stronger the movement.

To remove the proportionality, we introduce a constant: the *diffusion coefficient*, $D$, which tells us how easily particles spread. This gives us:

$$\textbf{\textit{Flux}} = \textbf{\textit{-D}} \times \textbf{\textit{Change in concentration over distance}}$$

## *So what is flux?*

Think of a faucet. If you barely open it, only a small drop comes out. If you open it fully, a lot of water flows out every second. The amount of fluid passing through an area per second is called flux. A lower flux means less water is flowing, while a higher flux means more water is flowing. In the same way, in diffusion, flux tells us how much of a substance moves through a certain area over time.

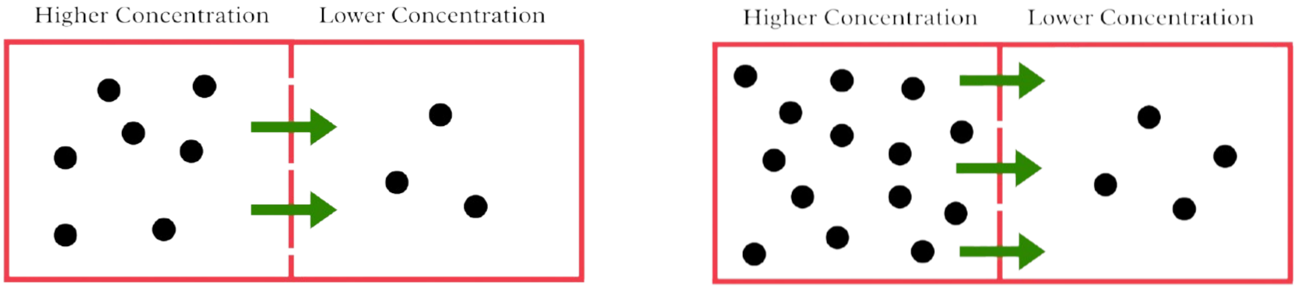
Figure 4: A slightly open faucet with low flux (less water flowing per second).

Figure 5: A fully open faucet with high flux (more water flowing per second).

## *But what causes this movement?*

This brings us to the idea of a gradient. A gradient simply means how much something changes over a distance. Imagine a hill. If it's steep, rolling a ball down will be fast; if it's nearly flat, the ball will barely move. The steeper the hill, the stronger the effect.

In diffusion, the concentration gradient is like that hill. If one side has a much higher concentration than the other, the "slope" is steep, and particles will move quickly. If the concentration is nearly the same everywhere, movement slows down. This is why diffusion happens faster when there is a bigger difference in concentration over a shorter distance.



Low concentration gradient

High concentration gradient

Figure 6: The rate of diffusion would be greater in the second figure due to a higher concentration gradient.

[**Q 1a.**] Why do you think we introduced a negative (-) sign with our diffusion coefficient in Fick's first law?

[**Q 1b.**] Using dimensional analysis, determine the units of the *diffusion coefficient*, $D$. What do these units suggest about the nature of diffusion?

## Fick’s Second Law: How does concentration change over time?

If diffusion is driven by concentration differences, what happens as those differences start to disappear? At first, when one area has a much higher concentration than another, diffusion happens quickly. But as particles spread out, the difference becomes smaller, and diffusion slows down. Eventually, everything reaches a balance.

We can describe this as:

***Change in concentration over time*** $\propto$ ***- Change in flux over distance***

This means that as particles move, the concentration doesn’t just shift—it also **evolves over time**.

***We now know what flux is, but what’s change in flux over distance?***

Imagine a street where cars are entering and exiting at different points. On one end, cars are entering quickly, with many arriving per second. On the other end, cars are exiting slowly, with only a few leaving per second. Because the rate at which cars enter and exit changes along the street, something interesting happens: cars start building up in the middle, creating a traffic jam, while other areas remain less crowded.

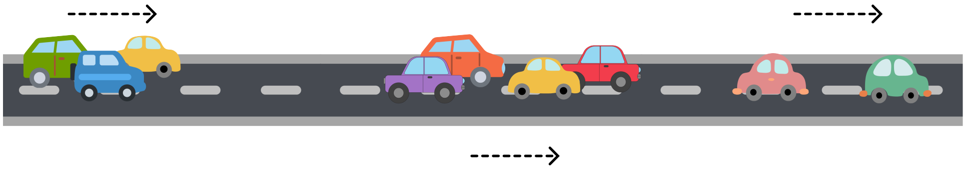


Figure 7: Changing flux (cars passing per second) affects concentration over distance and time.

This is exactly what happens in diffusion. If flux is higher in one region than in another, the concentration does not just decrease uniformly; it shifts at different rates in different places. If more particles are entering than leaving, concentration increases. If more are leaving than entering, it decreases. Over time, this difference smooths out, balancing the overall distribution.

While flux tells us how much is moving, change in flux over space tells us where particles are accumulating or depleting.

Fick’s Second Law describes this process: when flux changes across space, concentration doesn’t stay still—it evolves over time, continuously adjusting until things become more uniform.

[**Q** 1c.] Some substances diffuse faster than others even with the same gradient. What other properties may affect how easily a substance diffuses? Which part of Fick’s Laws may hold clues about these hidden properties?

Now that we know how diffusion works, we can finally test it. Let's move on to the setup. Remember, a well-prepared setup is key. It will help you capture important details and make the entire process smoother and more enjoyable!

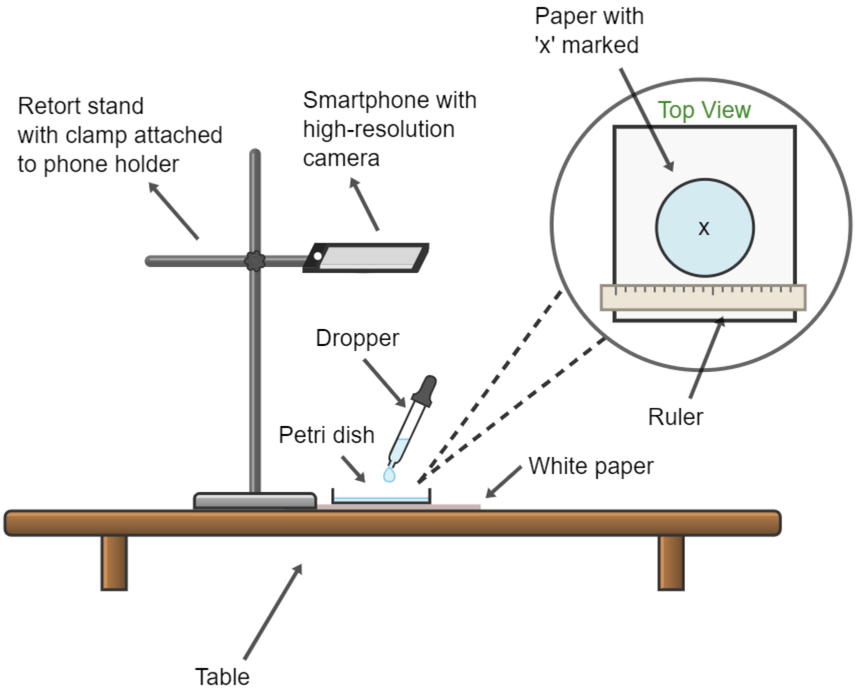


Figure 8: General set-up for the experiment.

1. Retort stand with clamp
2. Phone holder
3. Smartphone with high resolution camera
4. Petri dish
5. Ink
6. Dropper
7. White paper
8. Pencil
9. Ruler
10. Spare cloth
11. Small beakers
12. Vernier calipers
13. Gloves

## Procedure:

Start by setting up the equipment as shown in the diagram, making sure everything feels stable and secure. Pour enough water into the petri dish to create a smooth, even layer across the surface. On a sheet of paper, draw an 'x' in the center. This will help you keep everything nicely aligned as you work.

Attach the phone holder to the clamp, positioning it so it's parallel to the table. Place your smartphone in the holder and center the Petri dish over the 'x' on the paper. This way, your camera can capture the entire dish without missing any details. If your camera has gridlines, turn them on. They're super handy for alignment. Lay a ruler next to the dish so it's visible on the screen; it'll be your guide for measurements. Adjust the camera's height and zoom until the whole setup fits perfectly in the frame.

When everything is ready, hit record! Use a dropper to carefully place a single drop of ink on the water's surface. Hold the dropper close, but don't let it touch the water. Watch closely as the ink spreads across the surface.

## Video Analysis

Refer to Appendix A.1 for video analysis.

Now using the data collected, plot:

1. radius ($r$) vs. time ($t$)
2. radius ($r^2$) vs. time ($t$)

## What does the data say?

Now that we have plotted the data, let's take a closer look at what it tells us.

**[Q 1d.]** Do the two graphs look the same?
**[Q 1e.]** Does $r$ grow at a steady rate? What does the $r$ vs. $t$ graph show us?

### *Why do we plot $r^2$ instead of just $r$?*

The reason lies in the random motion of molecules. Molecules move in all directions: some outward, some sideways, and some even back toward the center. If we were to track individual molecules, their movements would cancel out when averaged, making it difficult to see the true extent of diffusion.

اس کو ہم شماریات میں Mean Squared
Displacement (MSD) بھی کہتے ہیں۔

By squaring the displacement, we ensure that all movements, whether forward or backward, contribute positively, preventing cancellation. This provides a clearer picture of how diffusion progresses and allows us to extract meaningful patterns from the data.

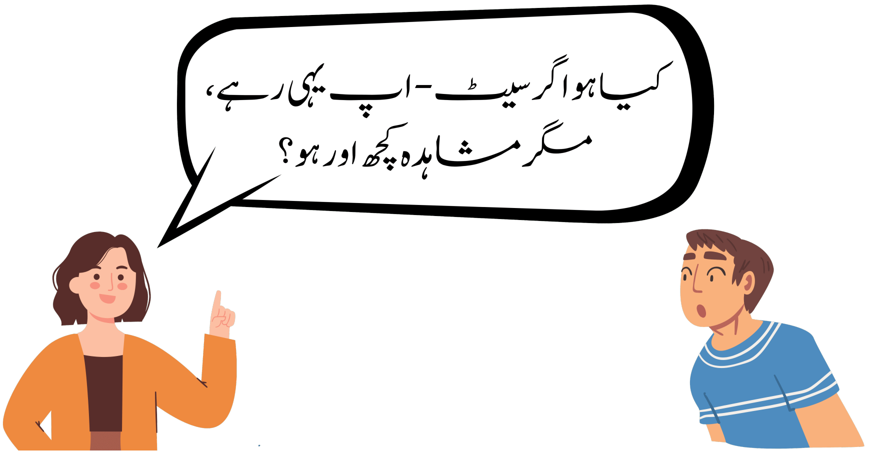
**[Q 1f.]** What is the relationship between $r^2$ and $t$? Does the graph look the same at each point? If not, what might be the reason?
**[Q 1g.]** Using dimensional analysis, determine the proportionality constant for this relationship.
**[Q 1h.]** Using the line of best fit, estimate its order of magnitude. How does this compare to typical values (check here: [1]) of your proportionality constant for diffusion in water? If there is a difference, what might explain it?

کیا آپ کو معلوم ہے کہ جو تعلق آپ نے
دریافت کیا وہ دراصل فک کے دوسرے قانون
(Fick's Second Law) کا ایک حل ہے

**[Q 1i.]** If diffusion slows down as concentration differences decrease, does this mean diffusion ever stops? Why or why not?

# 1b) Ink in Water: Marangoni Flows

### مزے کا مشاہدہ اور سوال



Before we move on to our experiment, let us quickly revisit the concept of surface tension. Molecules in a liquid exert forces on one another. Inside the liquid, the forces are balanced as molecules are surrounded on all sides. However, on the surface, the molecules experience an unbalanced force that pulls them inward. This creates a “film” on the surface. The tension in this film is what we call the surface tension. Each liquid has its own surface tension, which means that the molecules at the surface experience different magnitudes of force depending on the liquid.

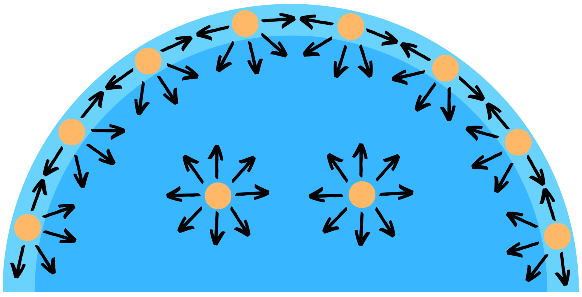


Figure 9: Net force on molecules at surface is inwards.

## Procedure:

**i.** Start by setting up the equipment exactly as before, ensuring everything is stable and in place. This time, fill the Petri dish with ***just enough*** water to cover the entire surface evenly.

Watch closely as the ink spreads across the surface. This is where the real magic begins!

[**Q 1j.**] Before you drop the ink in water, make a prediction of what will happen.
[**Q 1k.**] Did your observation align with prediction? Did the drop settle and diffuse or spread quickly?

If your droplet dispersed rapidly, what you just observed is called the **Marangoni effect**.

This phenomenon occurs due to surface tension gradients. When a drop of liquid with low surface tension (such as ink) falls on a thin film of liquid with higher surface tension (like water), a force is created. The higher surface tension of the water pulls on the ink, causing it to spread rapidly. According to Newton's law, an object moves when there is an unbalanced force acting on it, and it moves in the direction of the resultant force. In the case of the Marangoni effect, the ink moves in the direction where the surface tension is higher, resulting in the rapid dispersion of the ink.

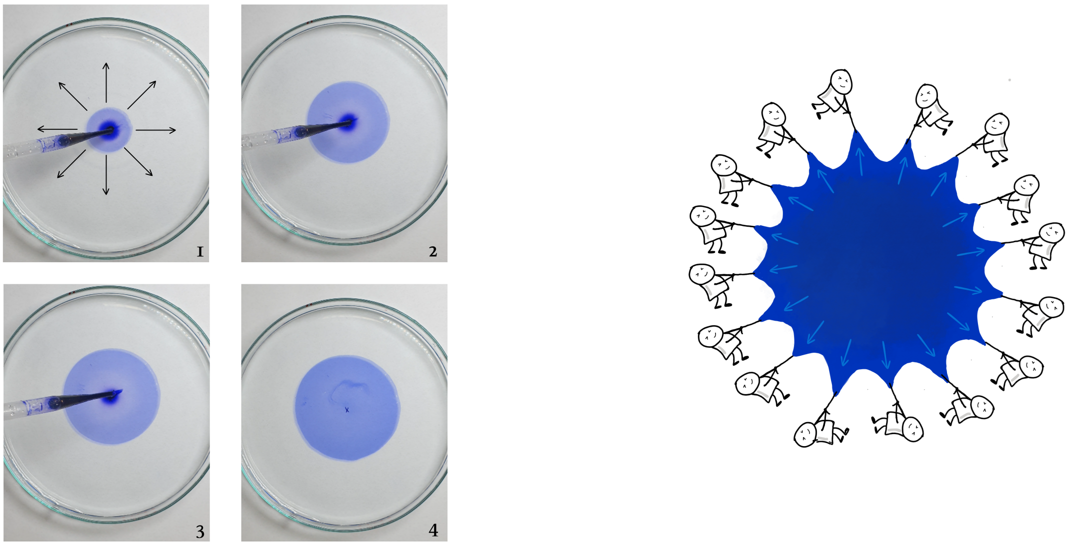


Figure 10: Imagine water molecules pulling ink from all sides as it travels from region of high surface tension to low surface tension [time scale: 1 s].

کیا آپ اندازہ لگا سکتے ہیں کہ کیوں آپ نے مارانگونی
تمَوُّج (Marangoni Effect) دیکھا نہ کہ نُفُوذ (diffusion)

Diffusion and Marangoni flows work in completely different ways and have their own unique behaviors. Diffusion happens because of concentration differences. Think of it as fluid moving naturally from where it's crowded (high concentration) to where there's more space (low concentration). This slow and steady spread is also helped along by the random jiggling of molecules, called Brownian motion.

Marangoni flows, however, are all about surface tension. Changes in surface tension—caused by things like temperature, evaporation, or impurities—create a kind of tug-of-war on the surface. This results in quick, directed motion, making the ink zip around in surprising ways.

Which takes charge, diffusion or Marangoni, depends on the conditions. If concentration differences are stronger, diffusion leads to a calm, even spread. But when surface tension differences are bigger, Marangoni flows create rapid and sometimes chaotic motion. Even tiny changes in these forces can tip the balance, creating a completely different outcome!

[Q 1l.] Did the ink continue to spread like it did during diffusion? If not, can you guess why?

**ii.** Now, let's add a little twist to the experiment by introducing soap as a surfactant. Surfactants, like soap, are special because they lower the surface tension of liquids. Soap molecules have two parts: a water-loving (hydrophilic) head and a water-fearing (hydrophobic) tail. This unique structure helps soap break the strong bonds between water molecules, making the surface tension weaker and allowing the liquid to spread more easily.

اشارہ: الکحل کی تبخیر

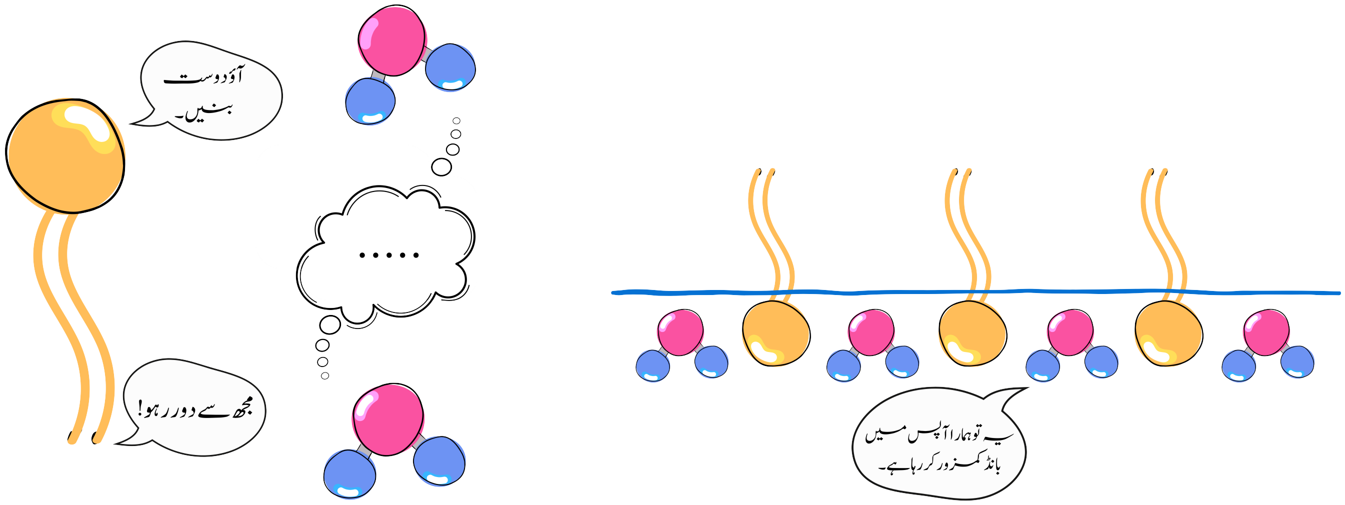


Figure 11: Surfactant molecules and their effect on surface tension.

When you add soap to water, it changes the game. Substances like ink will behave differently on the surface, creating new and exciting patterns to observe! ***Protip: Use pepper flakes for better visibility!***

کیا ہوگا اگر ہم سیاہی کی جگہ الکحل یا ایسی ٹون استعمال کریں

Additional apparatus required:

1. Isopropyl alcohol
2. Acetone
3. Safety goggles
4. Mask

[Q 1m.] How does the alchohol behave when you add it to water?
[Q 1n.] How does the spread of the alcohol/acetone compare to when you used ink or soap?
[Q 1o.] Why do you think the spread is different in alcohol? What could the surface tension and other properties of alcohol have to do with this?

# 2) Marangoni Bursting

Have you ever seen fireworks? What if we told you that we could create something similar, but instead of sparks in the sky, they happen in oil? In our previous experiments, we observed flows in miscible fluids: liquids that mix together easily. But now, we're about to try something new with immiscible fluids: those that don't mix at all, like oil and water.

To start, let's understand why some fluids don't mix. Take water and oil as an example. Water is polar, meaning it has a positive and negative side, which allows the molecules to form strong hydrogen bonds with each other. Oil, on the other hand, is non-polar, so its molecules do not have a positive or negative side and interact weakly with each other.

کیا آپ جانتے ہیں کہ شیرہ سازی
(emulsification) کیا ہے؟ آپ اسے اپنی
روزمرہ زندگی میں کہاں دیکھتے ہیں

When you try to mix them, the water molecules stick to each other strongly, and the oil molecules do the same. The attraction between water molecules is much stronger than the interaction between water and oil, so the two do not mix.

Before you try mixing, you might think they'll blend together, but in reality, they'll stay separate. This happens because the strong hydrogen bonds in water hold the molecules together, while oil molecules are held together by weaker forces. So, when water and oil are combined, they form distinct layers.

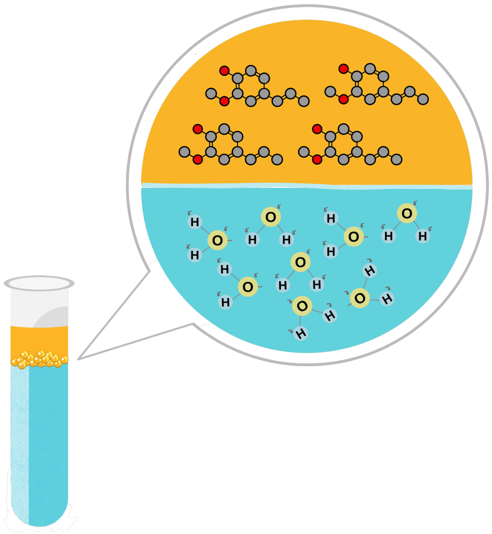


Figure 12: Water's polar and oil's non-polar molecules forming distinct layers.

CAUTION

Do not pour oil down the drain. It can clog pipes. Dispose of oil and chemicals properly.

Let us move onto our experiment. For that, we need the following:

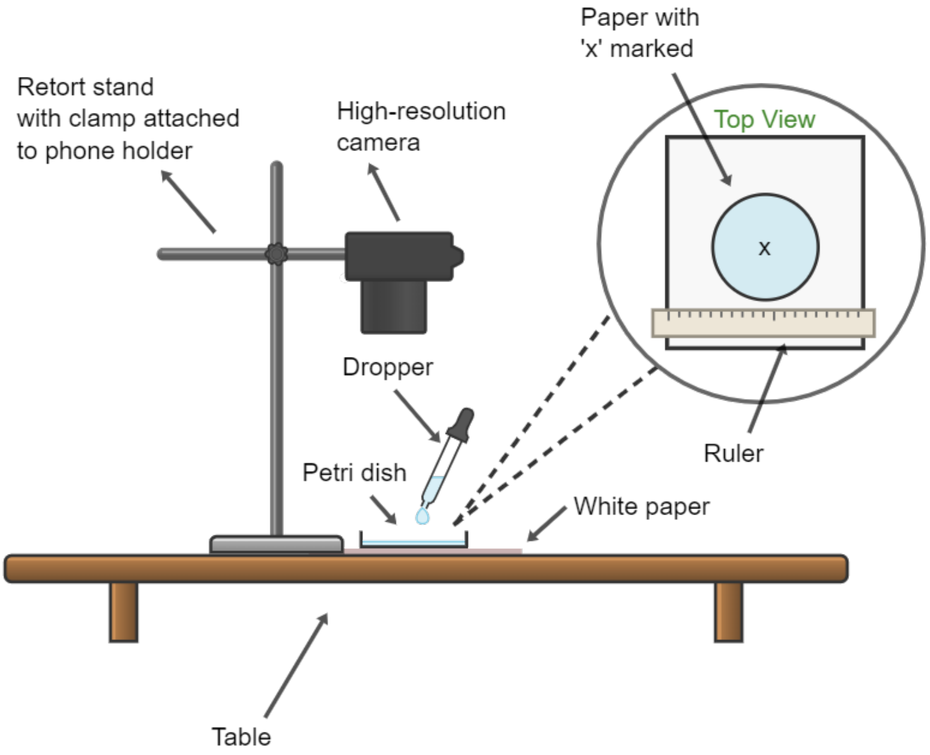


Figure 13: General set-up for the experiment.

1. Retort stand with clamp
2. High-resolution camera
3. Petri dish
4. Sunflower oil
5. Water
6. Isopropyl alcohol
7. Balance
8. Food coloring/Methylene blue dye
9. Dropper
10. White paper
11. Pencil
12. Ruler
13. Spare cloth
14. Small beakers
15. Vernier calipers
16. Gloves
17. Mask

## Procedure:

Start by setting up the equipment just like before, making sure everything is stable and well-positioned. This time, you'll be using a DSLR camera, so mount it securely on a clamp, keeping it level and directly above the Petri dish. Adjust the focus and zoom until the entire dish fits neatly in the frame.

Pour 0.8–1 cm of oil into the Petri dish, creating a smooth, even layer. Place it precisely on the marked spot to keep everything aligned. Prepare your IPA-water mixtures with different mass fractions, carefully measuring each using a balance. Add a small amount of dye to the mixture for better visibility. Once everything is ready, start recording. Use a dropper to release a droplet of the IPA-water mixture into the oil and observe how it interacts with the surface.

These brilliant "fireworks" are another manifestation of the Marangoni flows. We now know that Marangoni flows occur when there is a difference in surface tension between two fluids. These surface tension gradients can be due to temperature variations or concentration variations. Evaporation can cause changes to both of these.

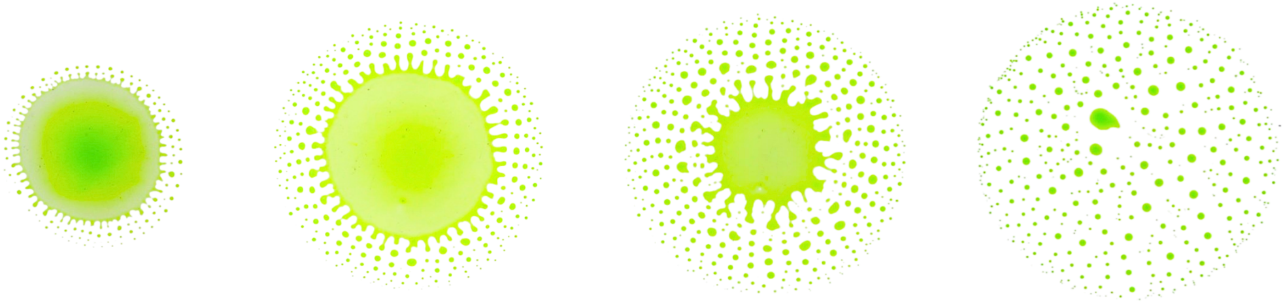


Figure 14: The droplet undergoes Marangoni bursting, breaking into smaller droplets due to surface tension gradients [time scale: 5 s].

### *What changed when we added alcohol to water?*

Alcohol, or isopropyl alcohol, is special because it can mix with both water and oil. It has a hydroxyl group (-OH) that makes it polar, allowing it to bond with water and other polar substances. At the same time, its non-polar hydrocarbon chain can interact with oils. This dual nature changes how water behaves, helping it mix with substances it normally wouldn't.

When alcohol is added to water, it reduces the strong forces between water molecules, lowering the surface tension. Alcohol also evaporates faster than water, and this happens most quickly at the edges of a droplet. As a result, there is less alcohol at the edges than in the middle. Since alcohol lowers the overall surface tension, the middle of the droplet, where alcohol concentration is higher, has a lower surface tension than the outer region. This difference creates an outward flow of liquid from the center to the edges.

As more alcohol evaporates, the edges become unstable. This instability can cause the droplet to break into smaller droplets. These new droplets initially contain both water and alcohol, but as alcohol continues to evaporate, they become mostly water. Because water does not mix with oil, these droplets remain separate, floating as tiny water spheres in the surrounding oil. This entire process—caused by surface tension differences, rapid evaporation, and immiscibility—leads to fascinating movement and patterns at the droplet's edges.

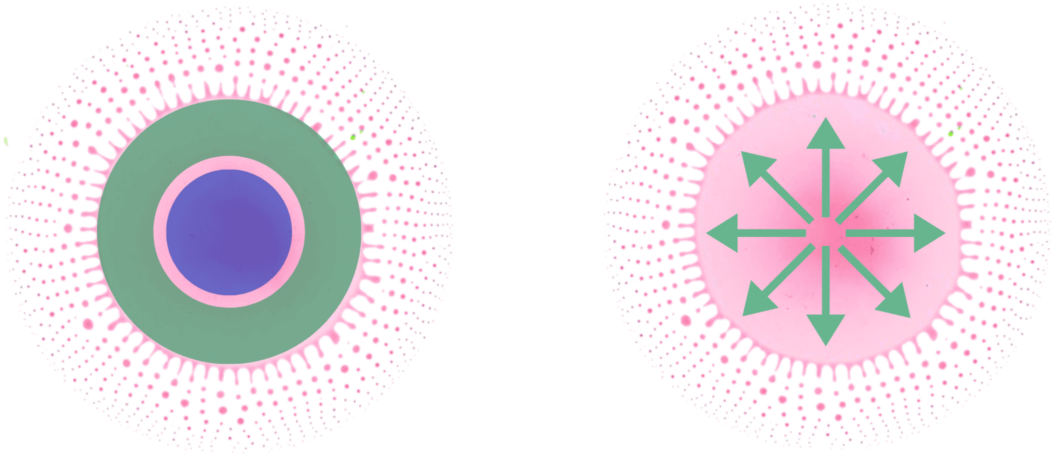


Figure 15: As alcohol evaporates faster at the edges, the middle (purple) holds more alcohol while the edges (green) have less, creating a surface tension gradient that pushes liquid outward.

## Video Analysis

Refer to Appendix A.1 for video analysis. The process for measuring radius is the same for all experiments.

Repeat the experiment for different **mass fractions of IPA** in the IPA-water mixture. You could try from 0.1-0.8 mass fractions.

Using the data collected, plot:

1. Mother droplet radius ($r_m$) vs. time ($t$)
2. Outer radius ($r_o$) vs. time ($t$)

Next, normalize both the **radius** and **time** by dividing each by its highest value in the dataset. Then, plot:

3. Normalized mother droplet radius ($r_{mn}$) vs. normalized time ($t_n$)
4. Normalized outer ($r_{on}$) radius vs. normalized time ($t_n$)

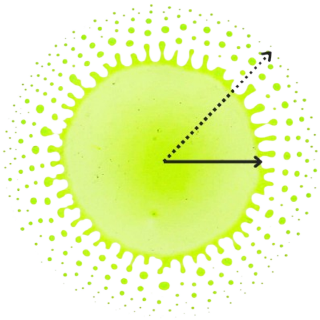


Figure 16: Solid line: mother droplet radius, dotted line: outer radius

## What does the data say?

Now that we have plotted the data, let's take a closer look at what it tells us.

**[Q 2a.]** At what threshold mass fraction of IPA does Marangoni bursting first appear? Can you identify a range where this transition occurs?

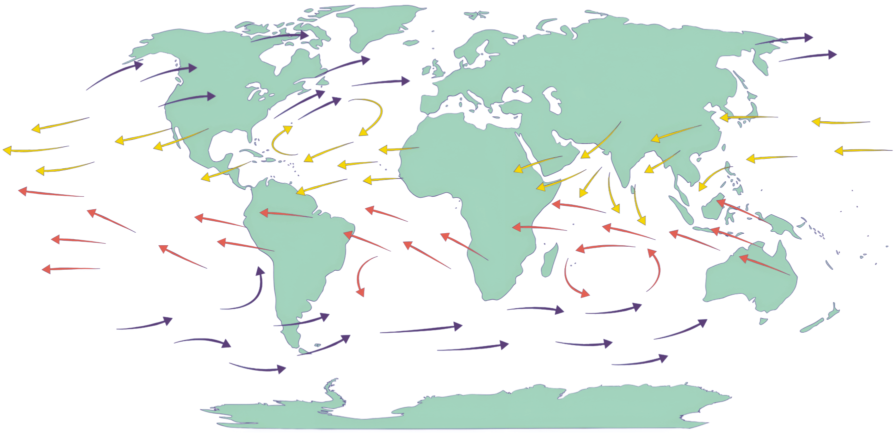
**[Q 2b.]** What can you deduce from ($r_m$) vs. ($t$) and ($r_o$) vs. ($t$) graphs?

**[Q 2c.]** Why do you think we normalize our quantities before plotting? How does this help in comparing different datasets?

**[Q 2d.]** What do you notice about the ($r_{mn}$) vs. ($t_n$) and ($r_{on}$) vs. ($t_n$) graphs? What does it tell us about our experiment?

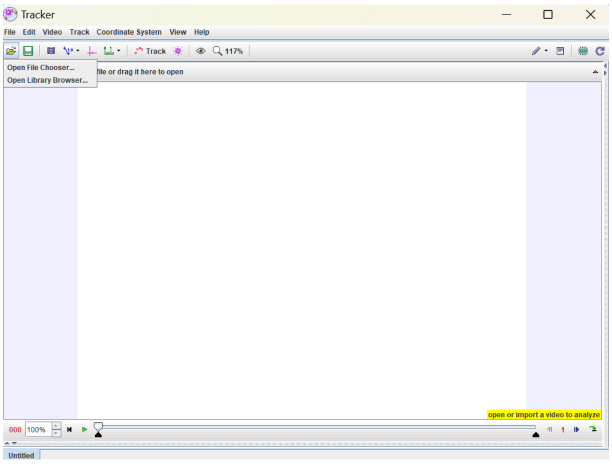
**[Q 2e.]** As you vary the mass fraction of IPA, do you notice any shifts in the overall flow or fragmentation pattern?

مزے کا سوال: آپ کے خیال میں ہمارے لیے سیالات کے بہاؤ کا مطالعہ کرنا کیوں ضروری ہے؟



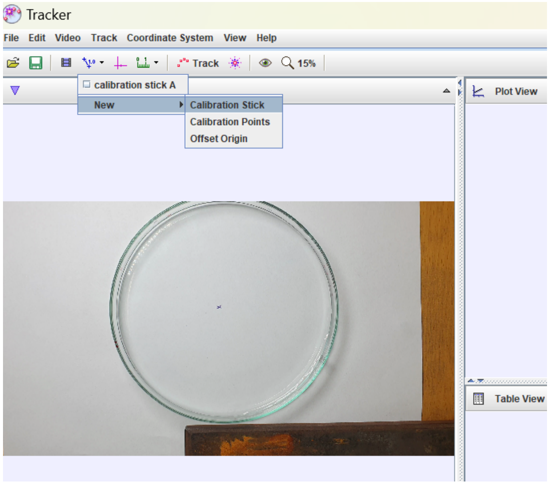
# A Appendix

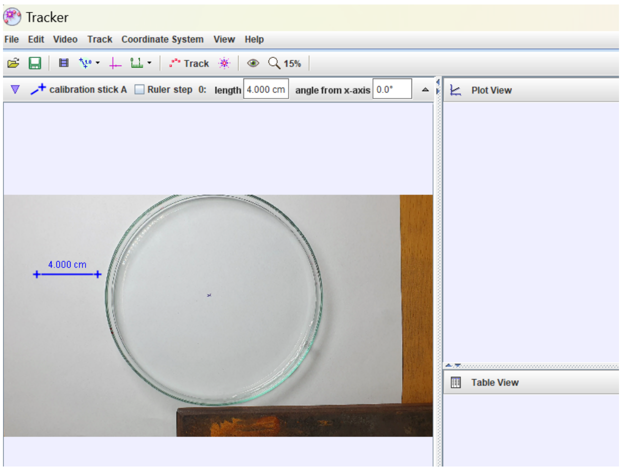
## A.1 Data analysis



Open the tracker app and click ***Open File Chooser*** to load the recorded video file.

Select ***Calibration Stick*** from the menu.

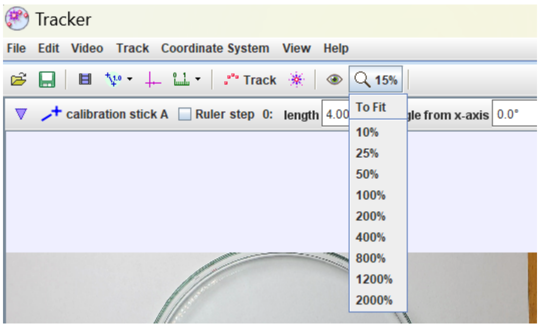


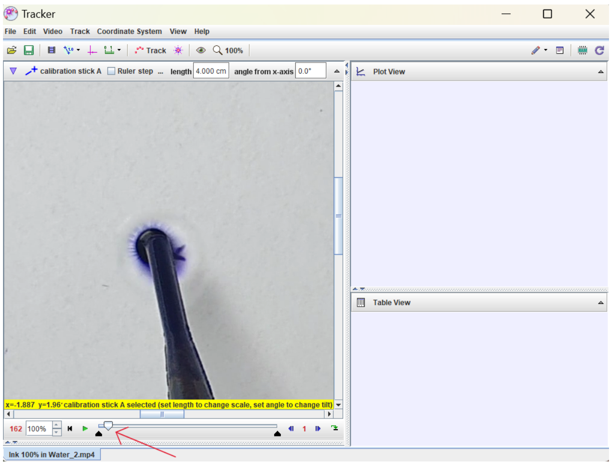


calibration stick A ☐ Ruler step 0: length 4.000 cm angle from x-axis 0.0°

Use the ruler in your video as a reference to calibrate the frame for accurate measurements.

Zoom in to ensure the droplet is clearly visible for precise marking.

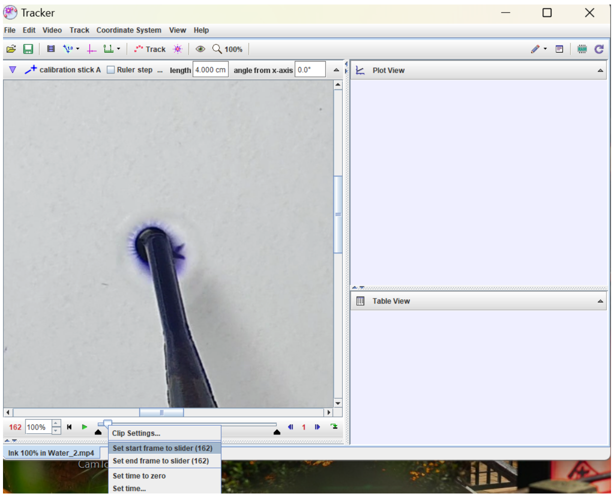


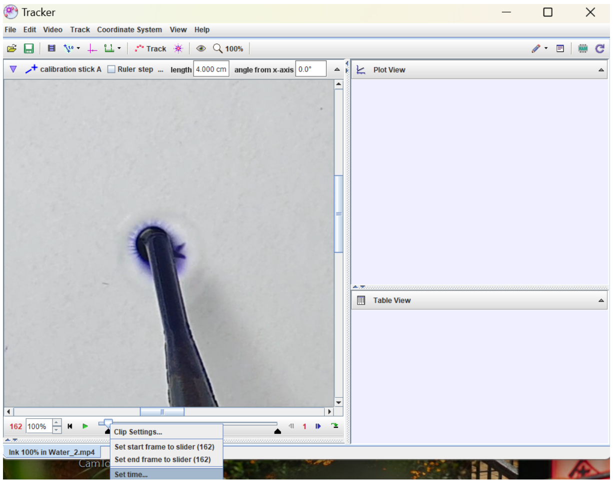


x=-1.887 y=1.96′ calibration stick A selected (set length to change scale, set angle to change tilt)

Move the ***Slider*** to the frame where the droplet leaves the dropper and first touches the surface of the liquid.

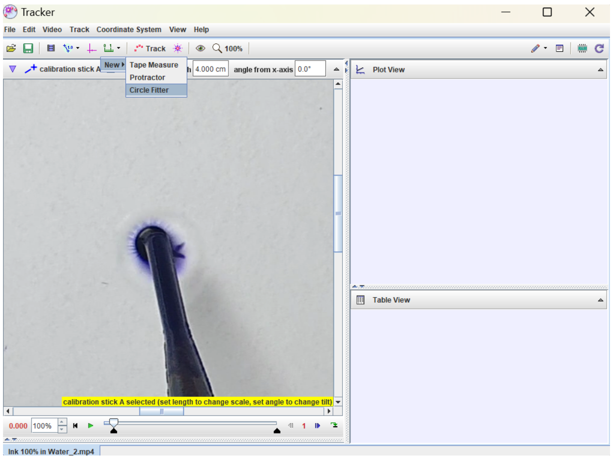
Set the ***Start Frame*** to this position.



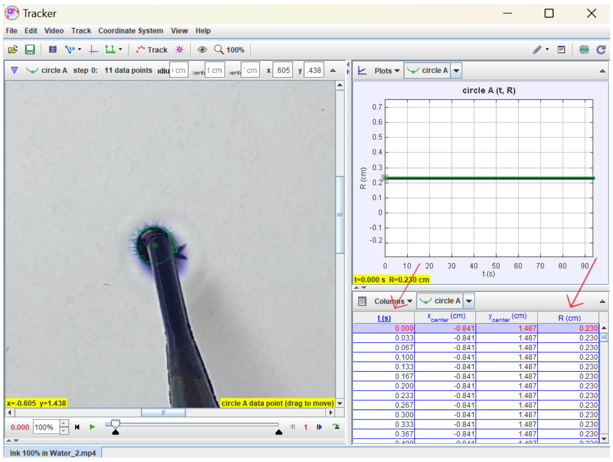


Reset the timer so that t = 0.000 s corresponds to the moment the droplet touches the surface.

Select ***Circle Fitter*** from the tools.



calibration stick A selected (set length to change scale, set angle to change tilt)



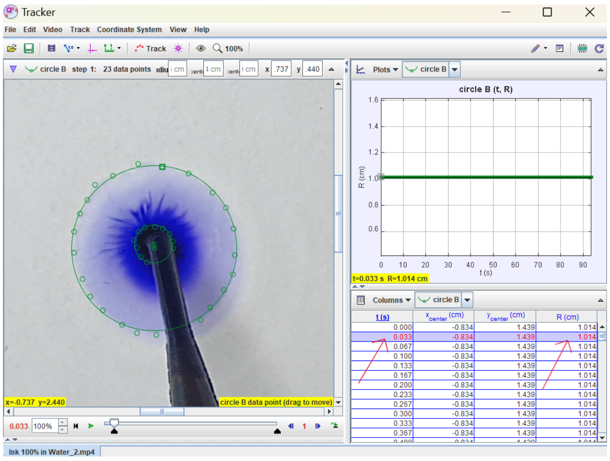
circle A data point (drag to move)

t=0.000 s R=0.230 cm

| t (s) | $x_{center}$ (cm) | $y_{center}$ (cm) | R (cm) |
|---|---|---|---|
| 0.000 | -0.841 | 1.487 | 0.230 |
| 0.033 | -0.841 | 1.487 | 0.230 |
| 0.067 | -0.841 | 1.487 | 0.230 |
| 0.100 | -0.841 | 1.487 | 0.230 |
| 0.133 | -0.841 | 1.487 | 0.230 |
| 0.167 | -0.841 | 1.487 | 0.230 |
| 0.200 | -0.841 | 1.487 | 0.230 |
| 0.233 | -0.841 | 1.487 | 0.230 |
| 0.267 | -0.841 | 1.487 | 0.230 |
| 0.300 | -0.841 | 1.487 | 0.230 |
| 0.333 | -0.841 | 1.487 | 0.230 |
| 0.367 | -0.841 | 1.487 | 0.230 |

Mark points along the perimeter of the droplet to fit a circle to its shape. The time and the calculated radius will appear on the right side of the screen. *(Note: This will provide an average radius of the drop.)*

Move to the next frame and repeat the process. Carefully note the time and radius for each frame.



circle B data point (drag to move)

t=0.033 s R=1.014 cm

| t (s) | $x_{center}$ (cm) | $y_{center}$ (cm) | R (cm) |
|---|---|---|---|
| 0.000 | -0.834 | 1.439 | 1.014 |
| 0.033 | -0.834 | 1.439 | 1.014 |
| 0.067 | -0.834 | 1.439 | 1.014 |
| 0.100 | -0.834 | 1.439 | 1.014 |
| 0.133 | -0.834 | 1.439 | 1.014 |
| 0.167 | -0.834 | 1.439 | 1.014 |
| 0.200 | -0.834 | 1.439 | 1.014 |
| 0.233 | -0.834 | 1.439 | 1.014 |
| 0.267 | -0.834 | 1.439 | 1.014 |
| 0.300 | -0.834 | 1.439 | 1.014 |
| 0.333 | -0.834 | 1.439 | 1.014 |
| 0.367 | -0.834 | 1.439 | 1.014 |

## A.2 Mathematical model for diffusion

The solution to the diffusion equation, also known as Fick's second law, in two dimensions is given by:

$$C(x,t) = \frac{M}{4\pi Dt} e^{-x^2/4Dt}$$

For radial coordinates, which are more relevant to our system, the solution takes the form:

$$C(r,t) = \frac{M}{4\pi Dt} e^{-r^2/4Dt}$$

**Note:** As an exercise, try verifying this solution by taking the time and radial derivatives and substituting them back into the original diffusion equation.

Let's see how we can achieve a solution like this. A good tip is to try the steps by hand because it often helps you understand things more clearly.

We want to solve the 2D radial diffusion equation:

$$\frac{\partial C}{\partial t} = D \left( \frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r} \right) \tag{1}$$

### Step 1: Separation of variables

Let's separate our function into two functions: one which only depends on radius, and the other which only depends on time. We will label these $R(r)$ and $T(t)$ respectively. This way, we get an equation that looks like:

$$C(r,t) = R(r)T(t). \tag{2}$$

Now let's substitute equation (2) into equation (1), we find:

$$\frac{\partial [R(r)T(t)]}{\partial t} = \frac{\partial^2 [R(r)T(t)]}{\partial r^2} + \frac{1}{r} \frac{\partial [R(r)T(t)]}{\partial r} \tag{3}$$

In partial derivatives, we differentiate our function with respect to the variable we are interested in. Any other variable is assumed to be constant (*This is a key difference between partial derivative* $\frac{\partial}{\partial r}$ *and total derivative* $\frac{d}{dr}$).
Let's solve the derivatives in steps for clarity:

$$\frac{\partial C}{\partial t} = R(r) \frac{dT}{dt}, \tag{4}$$

$$\frac{\partial C}{\partial r} = T(t) \frac{dR}{dr}, \tag{5}$$

$$\frac{\partial^2 C}{\partial r^2} = T(t) \frac{d^2 R}{dr^2} \tag{6}$$

The diffusion equation becomes:

$$R(r) \frac{dT}{dt} = DT(t) \left( \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) \tag{7}$$

Dividing both sides by $R(r)T(t)$ gives:

$$\frac{1}{T(t)} \frac{dT}{dt} = D \frac{1}{R(r)} \left( \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) \tag{8}$$

As you can see, the left-hand side now depends only on $t$ and the right-hand side depends only on $r$.

Both must be equal to a separation constant, which we denote by $-\lambda$. We now have the system:

$$\frac{1}{T}\frac{dT}{dt} = -\lambda, \tag{9}$$

$$D\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) = -\lambda R \tag{10}$$

***You may ask why our equation is equal to a constant. Let's understand this.***

Assume that $\lambda$ is a function of $r$ and $t$. Now if we differentiate equation (9) with respect to $r$, one side on the equation will be equal to zero while other will be equal to $\frac{d\lambda}{dr}$. Similarly, if we differentiate equation (10) with respect to $t$, one side of the equation will be equal to zero while the other will be equal to $\frac{d\lambda}{dt}$. Now this is only possible if $\lambda$ is a constant, since we know that the derivative of a constant is always 0.

### Step 2: Solve the time part

Solving equation (9):

$$\frac{dT}{dt} = -\lambda T, \tag{11}$$

The solution is expected to be an exponential function, since the concentration profile naturally decreases over time as the system diffuses towards equilibrium:

$$T(t) = T_0 e^{-\lambda t}, \tag{12}$$

where $T_0$ is a constant.

### Step 3: Solve the radial part

Now solve the radial part, we write equation (10) like this:

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} = -\frac{\lambda R(r)}{D} \tag{13}$$

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \frac{\lambda R(r)}{D} = 0 \tag{14}$$

Multiplying both sides with $r^2$ gives:

$$r^2\frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} + r^2\frac{\lambda R(r)}{D} = 0 \tag{15}$$

This equation is in the form of Bessel function of order zero, and has the solution:

$$R(r) = C_1\, J_0(\sqrt{\frac{\lambda}{D}}\, r) + C_2\, Y_0(\sqrt{\frac{\lambda}{D}}\, r) \tag{16}$$

where $J_0$ and $Y_0$ are Bessel function of first kind of order 0 and Bessel function of second kind of order 0 respectively.

### Step 4: Combine the radial and time parts

Since we know that $C(r,t) = R(r)T(t)$, we get a solution like:

$$C(r,t) = [C_1\, J_0(\sqrt{\frac{\lambda}{D}}\, r) + C_2\, Y_0(\sqrt{\frac{\lambda}{D}}\, r)]\, T_0 e^{-\lambda t} \tag{17}$$

At t = 0, the solution of $Y_0 \to \infty$. But we know that our concentration profile does not go to infinity. So, to keep our solution finite, the second term must be zero. Since $Y_0$ can not be zero, the constant $C_2$ must be. Therefore, we are left with:

$$C(r,t) = [C_1\, J_0(\sqrt{\frac{\lambda}{D}}\, r)]\, T_0 e^{-\lambda t} \tag{18}$$

Since $C_0$ and $T_0$ are constants, their product will also be a constant which we can represent by A to give:

$$C(r,t) = A\, J_0(\sqrt{\frac{\lambda}{D}}\, r)\, e^{-\lambda t} \tag{19}$$

### Step 5: Integrate to find the full solution

A mode is one basic solution of the equation that works for a specific value of $\lambda$. When we use separation of variables, we break the full solution into smaller pieces. Each mode is one of those pieces. The full solution is then made by adding up all the modes, using an integral:

$$C(r,t) = \int_0^\infty A(\lambda)\, J_0(\sqrt{\frac{\lambda}{D}}\, r)\, e^{-\lambda t}\, d\lambda \tag{20}$$

Here A is a function of $\lambda$. For each mode, $A(\lambda)$ gives a constant coefficient, but as we move across different values of $\lambda$, those constants might change. So, taken all together, A is a function of $\lambda$.

***Didn't we say that $\lambda$ is a constant earlier? How, then, are we differentiating with respect to it?***

It's true that we originally treated $\lambda$ as a constant when solving the equation by separating variables, and it is constant within each individual solution or mode. However, the full solution is built by adding together many such modes, each with its own fixed value of $\lambda$. So when we write the solution as an integral over $\lambda$, we're not saying $\lambda$ changes within a mode; we're just using it as a variable to sum over all the different modes. In that context, it's perfectly valid to treat $\lambda$ as a variable in the integral, even though it's constant within any single piece of the solution.

To describe diffusion, we use something called a delta function, $\delta(r)$, in physics. It means that at the point source at $r = 0$, all the mass is concentrated at that one point, and there's no mass anywhere else.

So how is it significant right now? It gives us the initial condition:

$$C(r,0) = M\delta(r)$$

$$\int_{allspace} C(r,0) = \int_0^\infty M\delta(r)\, 2\pi r = M \tag{21}$$

It's difficult to solve for $A(\lambda)$ directly from the delta function, so instead we make a simple assumption:

$$A(\lambda) = M$$

We can verify this assumption later on. Equation (20) then becomes:

$$C(r,t) = M \int_0^\infty J_0(\sqrt{\frac{\lambda}{D}}\,r)\, e^{-\lambda t}\, d\lambda \tag{22}$$

To solve this, let:

$$\mu = \sqrt{\frac{\lambda}{D}} \tag{23}$$

$$\lambda = \mu^2 D \tag{24}$$

$$d\lambda = 2D\, \mu d\mu \tag{25}$$

Substituting into equation (22) yields:

$$C(r,t) = 2MD \int_0^\infty \mu e^{-\mu^2 Dt}\, J_0(r\mu)\, d\mu \tag{26}$$

We can solve this using a known integral (given here: [2]):

$$\int_0^\infty x^{\nu+1} e^{-\alpha x^2} J_\nu(\beta x)\, dx = \frac{\beta^\nu}{(2\alpha)^{\nu+1}} e^{(-\frac{\beta^2}{4\alpha})}, \quad [\mathrm{Re}\alpha > 0,\ \mathrm{Re}\ \nu\ \ > -1] \tag{27}$$

Comparing equation (26) with equation (27) gives us the solution of the integral in the form:

$$C(r,t) = 2MD \cdot \frac{r^0}{(2Dt)^1} e^{-\frac{r^2}{4Dt}} \tag{28}$$

$$C(r,t) = \frac{M}{t} e^{-\frac{r^2}{4Dt}} \tag{29}$$

Now to verify our assumption, we take the limit as $t \to 0$:

$$\lim_{t \to 0} \frac{1}{t} e^{-\frac{r^2}{4Dt}} = \delta(r)$$

This is a well-known identity: as $t \to 0$, the given equation becomes infinitely narrow and tall, but its total area remains 1, which matches the definition of the delta function. So, using this result in our solution:

$$C(r,t) = \frac{M}{t} e^{-\frac{r^2}{4Dt}}$$

$$\lim_{t \to 0} C(r,t) = M\delta(r)$$

This confirms that our assumption $A(\lambda) = M$ correctly reproduces the initial condition $C(r,0) = M\delta(r)$.

### Step 6: Normalize the solution

We used $A(\lambda) = M$ inside the integral just to check if the solution has the correct shape. It gave us a function that becomes $M\delta(r)$ as $t \to 0$, which matches the initial condition. However, this does not guarantee that the total amount is exactly $M$. When we integrate the solution over all space, we get $4\pi D.M$, which is greater than what we started with. To fix this, we introduce a new constant $C_f$ in front of the solution. We choose $C_f$ so that the integral equals $M$. This step is called normalization. It makes sure that the solution represents the correct total amount of substance.

We start with the unnormalized solution in 2D:

$$C(r,t) = C_f . \frac{1}{t}\, e^{-\frac{r^2}{4Dt}} \tag{30}$$

We want to choose the constant $C_f$ so that the total concentration over all space is exactly M. In 2D, the area element is $2\pi r dr$, so we compute:

$$\int_0^\infty C(r,t).2\pi r\, dr = M \tag{31}$$

Substituting the expression for $C(r,t)$:

$$\int_0^\infty C_f.\frac{1}{t}\, e^{-\frac{r^2}{4Dt}}.2\pi r\, dr = M \tag{32}$$

Taking constants outside the integral:

$$\frac{2\pi C_f}{t}\int_0^\infty r\, e^{-\frac{r^2}{4Dt}}.dr = M \tag{33}$$

Let $u = \frac{r^2}{4Dt}$, so that $rdr = 2Dt\, du$. We get:

$$\frac{2\pi C_f}{t}.2Dt\int_0^\infty e^{-u}.du = M \tag{34}$$

$$4\pi D C_f = M \tag{35}$$

$$C_f = \frac{M}{4\pi D} \tag{36}$$

Thus, the final normalized solution becomes:

$$\boxed{C(r,t) = \frac{M}{4\pi Dt}\, e^{-\frac{r^2}{4Dt}}} \tag{37}$$

The general form of a Gaussian distribution in radial coordinates is:

$$f(r) = \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \tag{38}$$

Comparing this with the solution to the diffusion equation, we see that both share an exponential dependence on $r^2$, with diffusion time $t$ playing a role similar to the variance $\sigma^2$ in the Gaussian function. This results in bell-shaped concentration profiles that broaden over time as the ink spreads outward.

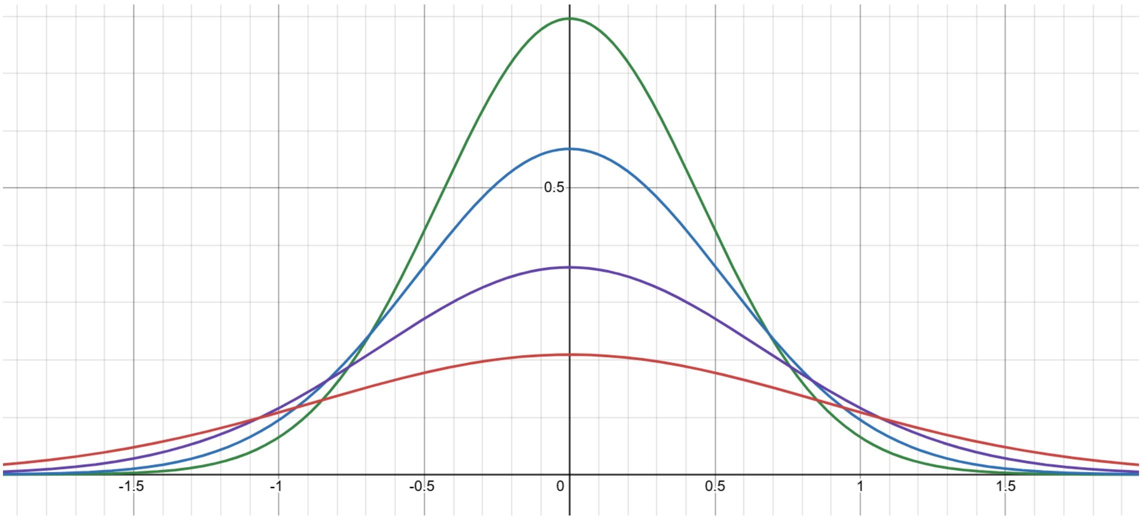


Figure 17: The concentration forms a bell-shaped curve at each time step. As time passes (green to blue to purple to red), the curve spreads out, showing that the particles are moving away from the point source.

The standard deviation of our bell-shaped curve is given as:

$$\sigma^2 = 2Dt$$

By comparing equations (37) and (38), we also confirm our result:

$$\langle r^2 \rangle = 4Dt$$

where $\langle r^2 \rangle$ is the mean squared displacement.

***You can find more details on the solution here:*** [1, 3, 4].

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